Original scientific paper

UDC:
159.955.2-053.2
159.922.72
37.091.113:51

Received: March, 04, 2026.
Revised: August, 12, 2026.
Accepted: August, 14, 2026.

10.23947/2334-8496-2026-14-2-317-329

Check for updates

Representational Difficulty Across Cognitive Stages in Quadratic Functions: An APOS–Newman Analysis

Hesty Marwani Siregar[1], *Turmudi*[1*], *Nurjanah*[1], *Darhim*[1]

[1] Universitas Pendidikan Indonesia – Faculty of Mathematics and Science Education, Bandung, Indonesia,
e-mail: hestymarwanisrg@upi.edu, turmudi@upi.edu, nurjanah@upi.edu, darhim@upi.edu

Abstract: This study aims to identify the types of students' mathematical representation errors using Newman's error classification and to analyze their relationships with the stages of cognitive development in the Action, Process, Object, and Schema theory. The study employed a qualitative descriptive approach involving 42 tenth-grade students from a senior high school. Data were collected through a mathematical representation ability test and in-depth interviews, and were analyzed qualitatively by mapping the types of errors and students' cognitive development stages. The results indicate that students' mathematical representation ability remains low, with symbolic representation the most challenging aspect across all ability levels. Comprehension and transformation errors were dominant among low-ability students, most of whom were at the Action stage. Moderate-ability students showed an emerging uneven pattern of cognitive attainment, with Schema-level attainment in the visual representation and, to a limited extent, in the verbal representation, whereas no Schema attainment was observed in the symbolic representation. Meanwhile, high-ability students demonstrated Schema-level attainment in the visual and verbal representations, whereas their symbolic performance remained at the Object stage because of persistent process-skill difficulties. These findings indicate that cognitive development is specific to the type of representation and does not occur uniformly across representations. This pattern of uneven cognitive attainment across representational forms is conceptualized in this study as asymmetrical representational development. The implications of this study suggest that the teaching of quadratic functions should emphasize coordination among visual, symbolic, and contextual representations through multi-representation approaches and gradual scaffolding to consistently support students' conceptual understanding.

Keywords: *mathematical representation, quadratic function, APOS, Newman error analysis, cognitive stages, asymmetrical representational development.*

Introduction

Mathematical representation ability constitutes a fundamental foundation in the construction of mathematical knowledge, as it enables students to coordinate symbolic expressions, visual structures, and contextual interpretations within a coherent conceptual system. From a cognitive perspective, mathematical processes invariably involve transformations across representations (Mainali, 2021). The involvement of multiple representational transformations suggests that conceptual understanding is constructed not only through procedural mastery but also through the ability to transform and integrate multiple systems of representation flexibly. The ability to transition from intuitive or visual representations to formal, analytical reasoning is an essential component in developing a more structured mathematical understanding (Santos-Trigo, 2024).

One mathematical topic that requires flexibility in representational ability is quadratic functions. A comprehensive understanding of quadratic functions requires the ability to transform symbolic expressions into graphical and other forms of representation (Tiew, Hoon, Singh, Walida, and Faradiba, 2023).

However, transforming between representations in the context of quadratic functions imposes a high

*Corresponding author: turmudi@upi.edu

cognitive load. Various studies indicate that students' difficulties do not always originate solely from procedural errors, but are related to obstacles in constructing and interpreting representations, particularly graphical representations derived from the symbolic form of quadratic functions (Luthfiani et al., 2025; Rahmawati et al., 2025; Wilkie, 2024).

Several previous studies have focused on identifying the types and frequencies of students' errors in solving quadratic function problems by employing error classifications such as Newman Error Analysis or categories of procedural and conceptual errors (Pramesti, Shoffa, and Soemantri, 2024; Rahman and Foad, 2021).

The Newman framework is primarily descriptive in identifying the location of errors and does not fully explain the cognitive and metacognitive mechanisms underlying them. In fact, transitions between representations require processes of exploration, meaning-making, and regulation of thinking that do not occur automatically (Faradiba, Sa'dijah, Parta, and Rahardjo, 2019; Fathonah, Cahyono, Haryani, Sarwi, and Lestari, 2024). These considerations suggest that error analysis alone is insufficient to understand students' thinking dynamics comprehensively.

APOS theory (Action, Process, Object, Schema), originally developed by Dubinsky, provides a cognitive framework for understanding how mathematical concepts are gradually constructed within students' mental structures. The action stage represents procedural activities that remain mechanical in nature; the process stage indicates the internalization of actions into mental operations; the object stage marks the ability to treat a process as a conceptual entity; and the schema stage reflects the organized integration of various conceptual objects within a more complex system of thought. APOS theory views mathematics learning as a process that proceeds through mechanisms of interiorization, coordination, and encapsulation until an organized conceptual schema is formed (Bintoro, Sukestiyarno, Mulyono, and Walid, 2021; Selden and Vidakovic, 2022; Sopamena, Kaliky, Sehuwaky, Kasliyanto, and Juhaevah, 2021).

Although APOS theory has been widely used to analyze students' mathematical understanding, most of these studies focus on characterizing conceptual structures and levels of understanding (Şefik, Uzun, and Dost, 2021). Empirical exploration of how representational errors develop as students transition from the action stage to the schema stage remains relatively limited, particularly in the context of quadratic functions. The integration of Newman's classification and APOS theory has the potential to provide a deeper understanding of students' representational errors. Through this integration, errors are no longer viewed merely as procedural mistakes, but as indicators of obstacles in understanding relationships among concepts and in coordinating different forms of representation.

Based on this framework, this study aims to identify the types of students' mathematical representation errors in quadratic function material using Newman's classification and to analyze how these errors relate to the stages of cognitive development in APOS theory. By linking students' error analysis with a cognitive development framework, this study seeks to explain how and why representational errors emerge and develop. Furthermore, the findings of this study are expected to assist teachers in designing learning activities that are more aligned with students' ways of thinking and their cognitive needs.

Materials and methods

This study employed a qualitative descriptive approach to analyze why students make errors in completing a mathematical representation ability test using the APOS theoretical framework. The approach was selected to provide an in-depth understanding of students' cognitive processes, particularly in revealing the stages of thinking that lead to errors based on the framework of Action, Process, Object, and Schema (APOS). Thus, this study aims not only to identify the types of errors students make but also to understand how their mental structures are formed when solving mathematical representation problems.

This study involved 42 tenth-grade students from one senior high school in the city of Pekanbaru as the research subjects. The school is a public educational institution that implements the Merdeka Curriculum and has taught quadratic functions. The criteria for students who became research subjects were that they had studied quadratic functions. The students participated voluntarily and provided written consent.

The implementation of this study was carried out through several main stages. The first stage was the preparation of research instruments, namely the development of mathematical representation ability test items that included three types of representation ability indicators, namely visual, symbolic, and verbal. These items then underwent an expert validation process to ensure the suitability of the content with the representation ability indicators being measured. After the validation process, the items were revised according to the suggestions from the validators, followed by a pilot test with a group of students outside the research sample to examine validity and reliability.

The second stage was administering the test to the research subjects. The test was given to 42 tenth-grade students at one senior high school in the city of Pekanbaru. Students were asked to solve the three representation problems independently within the allotted time. All student responses were collected for further analysis.

The third stage was scoring and grouping students by ability level. Each student's response was assessed against the preprepared scoring criteria. The obtained scores were then used to classify students into high, moderate, or low ability categories, using the mean and standard deviation as reference points. The criteria for grouping students' abilities are shown in the following table.

Table 1. *Criteria for grouping students' ability*

Score	Category
score < 3	Low
3 ≤ score ≤ 21	Moderate
score > 21	High

The fourth stage was the analysis of students' errors. Students' responses, which had been grouped, were analyzed to identify error types using the Newman error categories. The types of Newman errors used in this study included comprehension error, transformation error, process skill error, and encoding error. The description of error types and their indicators was carried out by modifying the criteria proposed by Siregar and Solfitri (2019) and adapting them to responses from the mathematical representation ability test on the quadratic function material.

Table 2. *Criteria for types of student errors using Newman theory*

No	Type of Error	Indicators
1	Comprehension	Leaving the answer blank Unable to identify the information required to answer the question Copying the question without continuing the solution
2	Transformation	Using an incorrect method in solve the problem Not transforming the information obtained in the problem into a form of representation that can solve the problem. Attempting to transform the information into a representation form, but errors occur in the representation provided
3	Process Skill	Making calculation errors Providing a problem-solving procedure but not completing it Providing a complete problem-solving procedure, but errors occur in the concept used
4	Encoding	Errors occur in writing mathematical symbols or notation Not writing the final answer Errors occur in writing the final answer

For verbal responses classified as encoding errors, interview data were used to further examine the source of the error. In cases where students obtained a mathematically correct result but were unable to express its meaning in the context of the problem, the underlying difficulty was interpreted as involving semantic translation or contextual mapping rather than notational production.

The fifth stage was the analysis of the APOS stages. At this stage, students' responses were also analyzed according to the Action, Process, Object, and Schema stages to determine which thinking stages had or had not been formed in each ability group. In this study, the criteria for grouping students' thinking stages based on APOS theory were developed by modifying those proposed by Yarman, Dwina, Murni, Yerizon, and Hevardani (2024) and adapting them to the quadratic function test instrument.

Table 3. *Criteria for students' thinking stages based on APOS theory*

No	APOS stage	Indicators
1	Action	Reading the given problem Identifying the information given in the problem and what is being asked Performing calculation operations in the steps of solving the problem Determining relevant coordinate points from the given quadratic function graph
2	Process	Determining the characteristics of a quadratic function Determining the vertex of a quadratic function using the formula Determining the y-intercept Determining the coordinate points passed by the quadratic function graph using substitution Drawing the quadratic function graph correctly based on the coordinate points given in the problem
3	Object	Drawing the graph accurately by considering the characteristics of the quadratic function graph Determining the value of coefficient b using the equation of the axis of symmetry Formulating the quadratic function equation Determining the maximum point of the quadratic function
4	Schema	Using the results of calculations (such as the vertex, roots, and graph) to understand the context of the problem given in the question Recognizing number patterns in the values of the quadratic function for various inputs Relating the maximum or minimum value of a quadratic function to practical applications in economic problems

The sixth stage was conducting interviews. To support the validity of the analysis results, unstructured interviews were conducted with students from each ability category: high, moderate, and low. The interviews aimed to explore more deeply the students' thinking processes when solving the problems, which were among the causes of the errors.

The final stage was data analysis and interpretation. Data from the tests and interviews were analyzed qualitatively to obtain a comprehensive description of the causes of students' errors in solving mathematical representation problems and their relationship with the stages of thinking in APOS theory.

Research data were collected through test and non-test techniques. The test technique involved administering a mathematical representation ability test. The mathematical representation ability test used included three indicators: visual, symbolic, and verbal. The non-test technique was conducted through interviews. The instrument used was an unstructured interview guide. The interview aimed to explore in greater depth the stages of students' thinking in solving representation problems, as well as to determine the reasons for these errors.

Data analysis was conducted in several stages. First, all student test responses were analyzed to identify errors based on the Newman categories. Data on the number of students who made each type of error were recorded and compared across ability categories.

Second, based on APOS theory, student responses were mapped to determine the stages achieved, starting with the action, process, and object stages, and progressing to the schema stage. These data were then compiled to observe the patterns of APOS stage achievement in each ability group and for each type of error.

Furthermore, interview data were analyzed to identify patterns of cognitive difficulties, misconceptions, and representational obstacles experienced by students. The results of this analysis were then triangulated with findings from the written tests to increase data credibility.

The validity of the data in this study refers to the trustworthiness criteria proposed by Lincoln and Guba, including credibility, transferability, dependability, and confirmability (Korstjens and Moser, 2018; Stahl and King, 2020). In this study, credibility was ensured through instrument validation by three mathematics education experts and method triangulation between written tests and interviews. Interviews were also used for member checking to ensure that the researcher's interpretation corresponded to the students' thinking. Transferability was addressed by providing descriptions of the school, student characteristics, and test material. Dependability was ensured through systematic documentation of the entire research process, forming an audit trail. Confirmability was conducted through validation of the analysis results by two mathematics teachers to ensure that the classification of strategies and errors was objective and consistent with classroom practice. The two teachers were not directly involved in the analysis, allowing them to provide independent judgments regarding the objectivity of the findings.

Results

The test results show that the average score for students' mathematical representation ability was 11.02 (26.89%), indicating that students' representation ability was still in the low category overall. Of the 42 students who became research subjects, 6 were categorized as the low-ability group, 27 as the moderate group, and 9 as the high-ability group.

Table 4. *Distribution of Students' Correct and Incorrect Answers*

Answer	Ability	Category					
		Visual		Symbolic		Verbal	
		n	%	n	%	n	%
Correct	Low	0	0	0	0	0	0
	Moderate	11	40.74	0	0	1	3.70
	High	1	11.11	0	0	5	55.56
Incorrect	Low	6	100	6	100	6	100
	Moderate	16	59.26	27	100	26	96.30
	High	8	88.89	9	100	4	44.44

Based on Table 4, which presents the representation indicators (visual, symbolic, and verbal), the symbolic indicator has the lowest success rate. This finding indicates that algebraic modeling and symbolic manipulation are the main challenges in understanding quadratic functions.

The analysis of errors based on Newman's classification for each representation indicator reveals distinct patterns for each representation type. The distribution is shown in Table 5 below.

Table 5. *Distribution of students' errors based on Newman classification*

Ability	Error Type	Category					
		Visual		Symbolic		Verbal	
		n	%	n	%	n	%
Low	Comprehension	4	66.67	1	16.67	5	83.33
	Transformation	1	16.67	5	83.33	1	16.67
	Process Skill	1	16.67	0	0	0	0
	Encoding	0	0	0	0	0	0
Moderate	Comprehension	6	22.22	7	25.93	8	29.63
	Transformation	4	14.81	10	37.04	12	44.44
	Process Skill	3	11.11	10	37.04	5	18.52
	Encoding	3	11.11	0	0	1	3.70
High	Comprehension	0	0	0	0	0	0
	Transformation	4	44.44	0	0	0	0
	Process Skill	3	33.33	9	100	1	11.11
	Encoding	1	11.11	0	0	3	33.33

For the visual indicator, students were asked to draw the graph of a quadratic function based on the provided verbal information. In this indicator, the difficulties of the low-ability group were dominated by difficulties in understanding and identifying the information needed to draw the graph (comprehension error). The moderate group showed a combination of comprehension and transformation errors, while the high-ability group experienced more transformation and process-skill errors in connecting verbal information to graphical representations.

Next, in the symbolic indicator, students were asked to construct the quadratic function formula based on information presented in the form of an image. This task required students to transform a visual representation into a symbolic form while performing appropriate algebraic manipulations. In general, errors in the symbolic indicator were concentrated at the transformation and process skill stages, although the distribution varied across ability levels.

In the low-ability group, most errors occurred at the transformation stage. Students were unable to convert the information contained in the image into an appropriate mathematical model. In the moderate-ability group, errors were distributed across two main categories: transformation and process skills. Students generally understood what needed to be done but experienced difficulties in constructing the function model accurately and performing algebraic calculations consistently. In the high-ability group, errors occurred more frequently at the process skill stage. Students generally succeeded in performing representation transformations correctly, but still made mistakes in calculation processes or in applying algebraic procedures. These errors indicate that at the high-ability level, obstacles no longer lie in understanding or transforming representations, but in the accuracy and consistency of symbolic manipulation.

In the verbal indicator, which tested students' understanding of the meaning of the vertex of a quadratic function in a real-world context, the low-ability group again showed dominance of comprehension errors, particularly in interpreting contextual information. Errors at the moderate-ability level mostly occurred at the transformation and process skill stages. They often experienced difficulties in converting problem information into mathematical form that could be solved and in applying problem-solving procedures correctly. Meanwhile, in the high-ability group, errors occurred more frequently at the encoding stage. Interview evidence indicated that, in some cases, students produced numerically correct results but were unable to express them in the contextual meaning required by the problem. These cases were interpreted as difficulties in semantic translation or contextual mapping rather than notational production.

Mapping the errors to the APOS stages shows a relatively consistent pattern of cognitive development. The distribution of errors by APOS stage for each indicator is shown in Tables 6, 7, and 8 below.

Table 6. *Distribution of APOS stages in visual problems*

Ability	Error type	Category							
		Action		Process		Object		Schema	
		n	%	n	%	n	%	n	%
Low	Comprehension	4	66.67	0	0	0	0	0	0
	Transformation	1	16.67	0	0	0	0	0	0
	Process Skill	0	0	1	16.67	0	0	0	0
Moderate	Comprehension	6	22.22	0	0	0	0	0	0
	Transformation	3	11.11	1	3.70	0	0	0	0
	Process Skill	0	0	2	7.41	1	3.7	0	0
	Encoding	0	0	0	0	3	11.11	0	0
High	Comprehension	0	0	0	0	0	0	0	0
	Transformation	2	22.22	2	22.22	0	0	0	0
	Process Skill	0	0	3	33.33	0	0	0	0
	Encoding	0	0	0	0	1	11.11	0	0

Table 7. *Distribution of APOS stages in symbolic problems*

Ability	Error type	Category							
		Action		Process		Object		Schema	
		n	%	n	%	n	%	n	%
Low	Comprehension	1	16.67	0	0	0	0	0	0
	Transformation	5	83.33	0	0	0	0	0	0
Moderate	Comprehension	7	25.93	0	0	0	0	0	0
	Transformation	10	37.04	0	0	0	0	0	0
	Process Skill	0	0	5	18.52	5	18.52	0	0
High	Process Skill	0	0	0	0	9	100	0	0

Table 8. *Distribution of APOS stages in verbal problems*

Ability	Error type	Category							
		Action		Process		Object		Schema	
		n	%	n	%	n	%	n	%
Low	Comprehension	5	83.33	0	0	0	0	0	0
	Transformation	1	16.67	0	0	0	0	0	0
Moderate	Comprehension	8	29.63	0	0	0	0	0	0
	Transformation	12	44.44	0	0	0	0	0	0
	Process Skill	0	0	2	7.41	3	11.11	0	0
	Encoding	0	0	0	0	1	3.70	0	0
High	Process Skill	0	0	1	11.11	0	0	0	0
	Encoding	0	0	0	0	1	11.11	2	22.22

To make the asymmetry between representation types explicit, Table 9 synthesizes the highest APOS stage attained by each ability group across the visual, symbolic, and verbal indicators, drawing on both the error-level mapping in Tables 6–8 and the correct-response data in Table 4. In constructing this synthesis, correct and complete responses were treated as evidence of Schema attainment, while erroneous responses were mapped to the highest APOS stage evidenced in the response; the full Newman error distributions are reported in Table 5, while their APOS-stage mappings are presented in Tables 6–8.

Table 9. *Cross-Representational Synthesis of APOS Attainment Across Ability Groups*

Ability	Visual	Symbolic	Verbal	Interpretation
Low	Action–Process; no Schema attained (comprehension dominant, n=4, 66.67%; no correct responses)	Action only; no Schema attained (transformation dominant, n=5, 83.33%; no correct responses)	Action only; no Schema attained (comprehension dominant, n=5, 83.33%; no correct responses)	Cognitive construction remained predominantly confined to the Action stage across representations, with only limited progression to Process in the visual representation and no Schema attainment observed.
Moderate	Schema attained (n=11, 40.74% correct); errors otherwise reached Object (n=4, 14.81%)	Action–Object; no Schema attained (highest error stage: Object, n=5, 18.52%; no correct responses)	Schema attained (n=1, 3.70% correct); errors otherwise reached Object (n=4, 14.81%)	Schema attainment emerged mainly in the visual representation and, to a much more limited extent, in the verbal representation, whereas symbolic performance remained below Schema, indicating emerging representational asymmetry and unstable coordination.
High	Schema attained (n=1, 11.11% correct); errors otherwise reached Object (n=1, 11.11%)	Object only; no Schema attained (all errors at Object, n=9, 100%; no correct responses)	Schema attained (n=5, 55.56% correct); 2 errors also occurred at Schema (22.22%)	Symbolic representation remained the critical bottleneck. Schema-level integration was achieved in visual and verbal representations but not in symbolic representation, providing the clearest evidence of asymmetrical representational development.

In the low-ability group, students' errors ranged from comprehension to process-skill errors, although comprehension and transformation errors were the most prominent. As summarized in Table 9, cognitive construction remained predominantly at the Action stage across the three representations, with only limited progression to the Process stage in the visual task. Students tended to rely on mechanical procedures without developing a sufficiently integrated understanding of the conceptual structure of quadratic functions. At this level, therefore, representational development remained relatively similar across visual, symbolic, and verbal tasks, with no evidence of Schema attainment.

A more differentiated pattern emerged in the moderate-ability group. Students' errors became more varied, ranging from comprehension to encoding errors, while erroneous responses spanned the Action to Object stages. At the same time, correct and complete responses evidenced Schema-level attainment in the visual representation and, to a much more limited extent, in the verbal representation, whereas no Schema attainment was observed in the symbolic task (Table 9). This pattern provides evidence of an emerging cross-representational asymmetry, as cognitive development did not progress uniformly across the three representation types. Although students had begun to internalize procedures and recognize relationships among solution steps, their coordination across representations remained unstable, with symbolic representation showing comparatively weaker development.

This cross-representational asymmetry became most pronounced in the high-ability group. No comprehension errors were identified, and Schema-level attainment was evident in both visual and verbal representations (Table 9). In contrast, no Schema attainment was observed in the symbolic representation, where all erroneous responses were mapped to the Object stage. This contrast identifies symbolic representation as the primary bottleneck in students' representational development and provides the clearest evidence of asymmetrical representational development. The presence of Schema-level attainment in visual and verbal representations did not correspond to equivalent attainment in symbolic representation, indicating that cognitive development was representation-specific rather than uniform across representational forms. Moreover, two erroneous responses in the verbal representation were mapped to the Schema stage (n=2, 22.22%; Table 8), indicating that Schema-level reasoning did not necessarily eliminate difficulties in expressing or contextualizing the final mathematical result.

The interview analysis shows differences in the characteristics of difficulties across students' ability levels. Examples of interview results are shown in Table 10 below.

Table 10. *Summary of examples of interview results*

Ability	Visual	Symbolic	Verbal
Low	In this problem, there are no coordinate points. So I followed the words in the question. When it says move upward, I draw a line upward. And when it says horizontal distance, I draw a horizontal line.	I don't really understand how to form a quadratic function formula from a graph, so I guess what the function might be.	I still don't really understand word problems, so it's difficult for me to answer them.
Moderate	I answer the problem by plotting the coordinate points given in the question and then connecting them with straight lines.	Actually, I struggle with problems that have graph images, so to answer the question, I immediately use the formula I remember.	I know that I should use the formulas to determine the maximum values of x and y, but I am not sure whether the problem asks for the values of the variables separately or as an ordered pair, so I did not continue the solution.
High	I tried to draw the coordinate points given in the problem, but I couldn't figure out how to continue the graph. I know the graph should be symmetrical.	I can obtain the function formula, but I don't really understand the pattern of the quadratic function values. It seems irregular.	I know that the maximum value is 225000, but if I have to explain its meaning in the economic problem, I'm not very sure because I'm not used to it.

Based on the interview results, low-ability students experienced difficulties at the initial stage of problem-solving. They showed confusion in understanding the information in the problem and in determining the initial steps, which led to not completing the problem or to guessing the answer. Moderate-ability students showed progress in following the instructions in the problem and recognizing relevant formulas. However, students in the moderate group still experienced confusion in selecting the appropriate formula or in continuing the problem-solving procedure to obtain the answer. Meanwhile, high-ability students did not experience difficulties in understanding the problem or using basic mathematical procedures. However, they showed confusion in interpreting calculation results in real-world contexts, particularly in distinguishing between mathematical and contextual meanings.

Discussions

The results of the study show that the average score for students' representations was 26.89%, indicating that students' mathematical representation ability remains generally low. However, the important finding of this study is not only the low average score, but also the presence of fairly clear differences between students' abilities in visual, symbolic, and verbal representations.

Based on Table 4, the symbolic indicator shows the lowest success rate across all ability groups, and there is no single correct answer. In contrast, in the visual and verbal indicators, although the success rate is still limited, the high-ability group shows relatively better performance. This pattern indicates that students' development of representation abilities is not evenly distributed across indicators.

The error analysis based on Newman's classification (Table 5) shows that in the symbolic indicator, errors are concentrated in the transformation and process skill categories at all ability levels. In the low-ability group, 83.33% of symbolic errors occur at the transformation stage. In the moderate group, errors are distributed between transformation and process skill. In the high-ability group, all symbolic errors occur at the process skill stage. This distribution indicates that the main obstacle in symbolic representation is not merely in understanding the initial information, but in the ability to transform visual representations into algebraic models, perform calculations and algebraic manipulation correctly, and determine the pattern of function values obtained.

Mapping symbolic errors onto the APOS stages (Table 7) further clarifies the cognitive structure underlying these difficulties. In the low-ability group, symbolic errors remained at the Action stage, indicating that students relied primarily on procedural actions without sufficiently internalizing the underlying relationships. In the moderate-ability group, erroneous responses extended from Action to Object, suggesting that some students had begun to internalize procedures and conceptualize quadratic functions more structurally, although this development remained inconsistent. In the high-ability group, all symbolic errors were mapped to the Object stage. Thus, although these students demonstrated a more advanced conceptual structure, they still experienced difficulties in carrying out symbolic processes accurately and consistently.

As synthesized in Table 9, this symbolic bottleneck becomes more apparent when APOS attainment is compared across the three representation types and ability groups. Schema-level attainment emerged in the visual and verbal representations among moderate- and high-ability students, whereas no Schema attainment was observed in the symbolic representation at any ability level. Instead, symbolic development remained within the Action-to-Object range, with Object representing the highest stage observed. This cross-representational contrast provides evidence of asymmetrical representational development, indicating that cognitive advancement in quadratic functions does not occur uniformly across representation types.

These findings demonstrate that the development of students' thinking in quadratic function material is representation-specific. This representation-specific development confirms that students may reach a higher stage in one form of representation, but this achievement is not necessarily stable in other forms of representation. In visual and verbal problems, moderate- and high-ability students are already able to understand the processes and concepts of quadratic functions quite well. However, students who have reached the Schema stage in visual and verbal representations still experience difficulties in symbolic representation. This pattern establishes that forming a stable Schema in quadratic function material requires mastery across multiple forms of representation rather than mastery of only one type. These results support the view that the ability to connect different forms of representation is very important in building deep mathematical understanding. Deep mathematical understanding cannot be constructed merely through mastery of a single form of representation, but through the ability to coordinate and connect various representations or models flexibly so that students can build a more comprehensive understanding and apply it in various contexts (Erath, Ingram, Moschkovich, and Prediger, 2021; Ke, Sadler, Zangori, and Friedrichsen, 2021; Kolar and Hodnik, 2021).

The interview results substantiate and extend this interpretation. In the symbolic indicator, a low-ability student stated, "I don't really understand how to make a quadratic function formula from a graph.

So I just guessed what the function might be." This statement exemplifies the Action-stage characteristic, in which they are not yet able to perform the transformation from graphical representation to algebraic form in a structured manner.

A moderate-ability student expressed, "Actually, I don't really like problems that involve graph images. So to answer the question, I use the formula that I remember." This statement shows that the student has remembered and used formulas but has not yet been able to connect visual information with symbolic models meaningfully. This response indicates that the procedure has been internalized, but the coordination between representations has not yet been strongly formed.

Meanwhile, a high-ability student stated, "I can obtain the function formula, but I don't know what the pattern of quadratic functions looks like." This quotation underscores a critical gap, although a symbolic model can be constructed, understanding of the structure of quadratic functions as a complete concept is not yet fully stable. In the verbal indicator, a high-ability student stated, "I know the maximum value is 225000, but if I have to explain its meaning in an economic problem, I am not very sure." The student's response was classified as an encoding error under Newman's framework because the final mathematical result was not expressed in the contextual form required by the problem. However, the interview evidence indicates that the underlying difficulty was not primarily notational; rather, it involved semantic translation or contextual mapping from a numerical result to its real-world meaning. Thus, successful symbolic calculation did not automatically lead to successful contextual interpretation.

Theoretically, these findings demonstrate that students' transition from the Action, Process, Object, to Schema stages does not always occur smoothly. Such difficulties in the early stages can hinder the development of a comprehensive understanding (Umam and Susandi, 2022). In addition, students' ability to understand and connect various forms of representation, such as graphs, symbols, and real contexts, strongly determines the strength of the conceptual understanding they construct (Altindis and Fonger, 2025; Li, Shen, Jiao, and Cai, 2022). These findings confirm that the type of representation used in a problem plays an important role in determining the stability of students' cognitive development within the APOS framework. Although APOS theory explains that understanding develops gradually from action toward more comprehensive understanding, this study reveals that such development does not always occur equally across all types of representation. These results extend previous research, which has generally examined APOS stages and mathematical representation abilities separately.

In this study, symbolic representation proved to be the most difficult part for students. Symbolic representation requires students to understand concepts, transform information into algebraic form, and perform calculations correctly. This combination of demands causes many students to experience greater difficulty in symbolic representation compared to other representations. These difficulties extend beyond the calculation stage, encompassing also students' ability to interpret and relate symbolic results to the problem's context. These results corroborate previous studies showing that students tend to experience difficulties when solving problems involving symbolic representations, both in translating them into verbal or visual forms and when connecting symbols with the conceptual meanings they represent (Hariyani, Suherman, Andriani, and Herawati, 2023; Nurrahmawati, Sa'dijah, Sudirman, and Muksar, 2021).

The encoding errors observed among high-ability students in verbal problems reveal that successful calculation does not necessarily ensure successful contextual interpretation. Interview evidence suggests that these errors were associated with difficulties in semantic translation or contextual mapping rather than merely with the notation or production of the final answer. This finding underscores that even when students can complete symbolic calculations, they may still struggle to explain the meaning of the results in real-world situations. These results confirm that connecting calculation results with contextual meaning requires additional abilities. The ability required is the ability to coordinate various forms of representation comprehensively. This ability relates to efforts to maintain semantic congruence across representations, which involves not only procedural manipulation but also coordination of meaning between symbolic and verbal representation systems (Ayala-Altamirano, Pinto, Molina, and Cañadas, 2022; Castro, Cañadas, Molina, and Rodríguez-Domingo, 2022).

The main contribution of this study lies in emphasising that difficulties in symbolic representation are not merely procedural problems, but are related to coordination between representations in the formation of conceptual objects and schemas. This finding extends the application of APOS theory in the context of multiple representations by showing that achieving certain cognitive stages does not necessarily hold across all forms of representation. Students may demonstrate better understanding in one representation, but still experience difficulties in other forms of representation.

These findings imply that learning quadratic functions should be designed to strengthen coordination among representations from the early stages. Learning approaches based on multiple representations and gradual scaffolding are believed to help students move more stably from the Action stage toward Schema. Several previous studies have found that learning interventions designed based on genetic decomposition within the APOS framework and combined with a multi-representation approach have been shown to facilitate coordination between representations and support the progression of students' mental construction from the Action stage toward Schema more systematically and stably (Baye, Ayele, and Wondimuneh, 2021; Borji, Martínez-Planell, and Trigueros, 2024; Siller, Nitzan-Tamar, and Kohen, 2023). This study extends previous findings by indicating that learning interventions need to consider that students at the Object stage still require support to integrate various forms of representation coherently until they reach the Schema stage.

This study is limited to one school context and uses a qualitative descriptive design, so the generalizability of the findings should be carefully considered. In addition, the analysis focuses on quadratic function material and does not directly test the effectiveness of learning interventions. Future research is recommended to develop experimental or quasi-experimental designs based on genetic decomposition within the APOS framework to test the effectiveness of multi-representation learning strategies empirically. Further studies may also examine the consistency of the cognitive development patterns identified in this study across different cultural and institutional contexts.

Conclusions

This study shows that difficulties in mathematical representation in the topic of quadratic functions are not only related to low procedural ability but are mainly associated with weak coordination among visual, symbolic, and verbal representations. Symbolic representation becomes the most challenging aspect for all ability groups. However, the main finding of this study is not merely the low scores achieved by students, but rather the different patterns of cognitive development across the forms of representation.

The integration of Newman's error classification with APOS theory reveals that certain types of errors are associated with students' stages of cognitive development. Low-ability students tended to remain predominantly at the Action stage, with comprehension and transformation errors being most prominent. Moderate-ability students exhibited an emerging uneven pattern of cognitive attainment across representations, with Schema attainment evident in the visual representation and, to a much more limited extent, in the verbal representation, whereas no Schema attainment was observed in the symbolic representation. This uneven pattern became most pronounced among high-ability students, who demonstrated Schema-level attainment in visual and verbal representations while symbolic performance remained at the Object stage.

These findings confirm that cognitive development in understanding quadratic functions is representation-specific rather than uniform across representational forms. We conceptualize this uneven pattern of APOS attainment as asymmetrical representational development, in which advancement to a higher cognitive stage in one representation does not necessarily correspond to equivalent attainment in another. Thus, this study extends the application of APOS theory in a multi-representation context by demonstrating that the formation of conceptual schemas is closely associated with students' ability to coordinate across representation systems.

Acknowledgements

The author gratefully acknowledges the scholarship support from the Lembaga Pengelola Dana Pendidikan (LPDP) / Indonesia Endowment Fund for Education under the Ministry of Finance of the Republic of Indonesia.

Funding

This work was supported by the Lembaga Pengelola Dana Pendidikan (LPDP) / Indonesia Endowment Fund for Education under the Ministry of Finance of the Republic of Indonesia.

Conflict of interests

The authors declare that there is no conflict of interest.

Data availability statement

The data that support the findings of this study are available from the corresponding author upon request. Restrictions apply due to ethical and privacy considerations.

Institutional Review Board Statement

This study was conducted in accordance with the ethical standards for research involving human participants and in line with the principles of the Declaration of Helsinki. The research protocol was reviewed and approved by the Ethics Committee of [Universitas Padjadjaran] (Approval Number: [1290/UN6.KEP/EC/2024], 27 December 2024).

Prior to participation, informed consent was obtained from all participants. Participants were informed about the purpose of the study, procedures, voluntary participation, and their right to withdraw at any time without consequences. Confidentiality and anonymity of participants were strictly maintained.

Author Contributions

Conceptualization, H.M.S. and T.; methodology, H.M.S., N., and D.; formal analysis, H.M.S.; investigation, H.M.S.; validation, T., N., and D.; supervision, T., N., and D.; writing—original draft preparation, H.M.S.; writing—review and editing, H.M.S., T., N., and D. All authors have read and agreed to the published version of the manuscript.

References

Altindis, N., & Fonger, N. L. (2025). A Culturally Inclusive Mathematics Learning Environment Framework: Supporting Students' Representational Fluency and Covariational Reasoning. *Education Sciences*, *15*(8), 980–1002. https://doi.org/10.3390/educsci15080980

Ayala-Altamirano, C., Pinto, E., Molina, M., & Cañadas, M. C. (2022). Interacting with Indeterminate Quantities through Arithmetic Word Problems: Tasks to Promote Algebraic Thinking at Elementary School. *Mathematics*, *10*(13), 2229–2246. https://doi.org/10.3390/math10132229

Baye, M. G., Ayele, M. A., & Wondimuneh, T. E. (2021). Implementing GeoGebra integrated with multi-teaching approaches guided by the APOS theory to enhance students' conceptual understanding of limit in Ethiopian Universities. *Heliyon*, *7*(5), 1–13. https://doi.org/10.1016/j.heliyon.2021.e07012

Bintoro, H. S., Sukestiyarno, Y. L., Mulyono, & Walid. (2021). The Spatial Thinking Process of the Field-Independent Students Based on Action-Process-Object-Schema Theory. *European Journal of Educational Research*, *10*(4), 1807–1823. https://doi.org/10.12973/EU-JER.10.4.1807

Borji, V., Martínez-Planell, R., & Trigueros, M. (2024). Students' Understanding of Riemann Sums and Double Integrals: The Case of Task Design in APOS Theory. *International Journal of Research in Undergraduate Mathematics Education*, 1–29. https://doi.org/10.1007/s40753-024-00250-6

Castro, E., Cañadas, M. C., Molina, M., & Rodríguez-Domingo, S. (2022). Difficulties in semantically congruent translation of verbally and symbolically represented algebraic statements. *Educational Studies in Mathematics*, *109*, 593–609. https://doi.org/10.1007/s10649-021-10088-3

Erath, K., Ingram, J., Moschkovich, J., & Prediger, S. (2021). Designing and enacting instruction that enhances language for mathematics learning: a review of the state of development and research. *ZDM – Mathematics Education*, *53*, 245–262. https://doi.org/10.1007/s11858-020-01213-2

Faradiba, S. S., Sa'dijah, C., Parta, I. N., & Rahardjo, S. (2019). Looking Without Seeing: The Role of Metacognitive Blindness of Student with High Math Anxiety. International Journal of Cognitive Research in Science, Engineering and Education (IJCRSEE), 7(2), 53–65. https://doi.org/10.5937/IJCRSEE1902053F

Fathonah, S., Cahyono, E., Haryani, S., Sarwi, S., & Lestari, N. H. (2024). Application of Multirepresentation-Based Creative Problem-Solving Learning Models to Improve Critical and Creative Thinking Skills for Students. *International Journal of Cognitive Research in Science, Engineering and Education (IJCRSEE)*, *12*(1), 185–200. https://doi.org/10.23947/2334-8496-2024-12-1-185-200

Hariyani, M., Suherman, Andriani, M., & Herawati. (2023). The Importance of Mathematical Representation Ability for Elementary School Students: A Literature Review and Its Implications. *Syekh Nurjati International Conference on Elementary Education (SICEE)*, *1*, 38–46. https://dx.doi.org/10.24235/sicee.v1i0.14579

Ke, L., Sadler, T. D., Zangori, L., & Friedrichsen, P. J. (2021). Developing and Using Multiple Models to Promote Scientific Literacy in the Context of Socio-Scientific Issues. *Science & Education*, *30*, 589–607. https://doi.org/10.1007/s11191-021-00206-1

Kolar, V. M., & Hodnik, T. (2021). Mathematical Literacy from the Perspective of Solving Contextual Problems. *European Journal of Educational Research*, *10*(1), 467–483. https://doi.org/10.12973/eu-jer.10.1.467

Korstjens, I., & Moser, A. (2018). Series: Practical guidance to qualitative research. Part 4: Trustworthiness and publishing. *European Journal of General Practice*, *24*(1), 120–124. https://doi.org/10.1080/13814788.2017.1375092

Li, S., Shen, Y., Jiao, X., & Cai, S. (2022). Using Augmented Reality to Enhance Students' Representational Fluency: The Case of Linear Functions. *Mathematics*, *10*(10), 1718–1738. https://doi.org/10.3390/math10101718

Luthfiani, U., Saragih, S., & Suanto, E. (2025). Development of learning based teaching materials on quadratic function materials to improve students' mathematical representation skills. *JTMT: Journal Tadris Matematika,* 6(2), 29–38. https://doi.org/10.47435/jtmt.v6i2.4191

Mainali, B. (2021). Representation in Teaching and Learning Mathematics. *International Journal of Education in Mathematics, Science, and Technology (IJEMST)*, *9*(1), 1–21. https://doi.org/10.46328/ijemst.1111

Nurrahmawati, Sa'dijah, C., Sudirman, & Muksar, M. (2021). Assessing Students' Errors in Mathematical Translation: From Symbolic to Verbal and Graphic Representations. *International Journal of Evaluation and Research in Education (IJERE)*, *10*(1), 115–125. https://doi.org/10.11591/ijere.v10i1.20819

Pramesti, C. A., Shoffa, S., & Soemantri, S. (2024). Analysis of Class X MAN Surabaya Students' Mistakes in Solving Function Questions Quadratics Based on Learning Styles and Newman Procedures. *JOELI: Journal of Educational and Learning Innovation*, *1*(2), 147–160. https://doi.org/10.72204/3e8ha171

Rahman, T. F. binti A., & Foad, M. S. bin M. (2021). Quadratic Functions in Additional Mathematics and Mathematics: An Analysis on Students' Errors. *Academic Journal of Business and Social Sciences*, *5*(1), 1–16.

Rahmawati, A. N. L., Sa'dijah, C., & Sudirman, S. (2025). Exploring high school students' mathematical representation abilities: Translation from verbal to symbolic using higher-order thinking skills. *TEM Journal,* 14(3), 2442–2453. https://doi.org/10.18421/TEM143-48

Santos-Trigo, M. (2024). Problem solving in mathematics education: tracing its foundations and current research-practice trends. *ZDM – Mathematics Education*, *56*, 211–222. https://doi.org/10.1007/s11858-024-01578-8

Şefik, Ö., Uzun, Ö. E., & Dost, Ş. (2021). Content Analysis of the APOS Theory Studies on Mathematics Education Conducted in Turkey and Internationally: A Meta-Synthesis Study. *Necatibey Faculty of Education Electronic Journal of Science and Mathematics Education,* 15(2), 404–428. https://doi.org/10.17522/balikesirnef.1020526

Selden, A., & Vidakovic, D. (2022). Remembering Ed Dubinsky and his Visionary Work. *International Journal of Research in Undergraduate Mathematics Education*, *8*, 421–436. https://doi.org/10.1007/s40753-022-00201-z

Siller, H., Nitzan-Tamar, O., & Kohen, Z. (2023). Scaffolding practices for modelling instruction in STEM-related contexts: insights from expert and novice teachers. *ZDM – Mathematics Education*, *55*, 1351–1364. https://doi.org/10.1007/s11858-023-01529-9

Siregar, H. M., & Solfitri, T. (2019). An Analysis of Students' Errors in Solving Indefinite Integral Problems Viewed From Gender Differences. *Journal of Research on Mathematics Instruction (JRMI)*, *1*(1), 17–24. https://doi.org/10.33578/jrmi.v1i1.12

Sopamena, P., Kaliky, S., Sehuwaky, N., Kasliyanto, & Juhaevah, F. (2021). Student Thinking Process in Solving Mathematical Problems Based on APOS. *Matematika Dan Pembelajaran,* 9(2), 31–46. http://dx.doi.org/10.33477/mp.v9i2.2384

Stahl, N. A., & King, J. R. (2020). Expanding Approaches for Research: Understanding and Using Trustworthiness in Qualitative Research. *Journal of Developmental Education*, *44*(1), 26–28.

Tiew, N. N. A. M. F., Hoon, T. S., Singh, P. S. A., Walida, S. El, & Faradiba, S. S. (2023). An Analysis of Students' Understanding of Algebraic Concepts. *Jurnal Pendidikan Sains Dan Matematik Malaysia*, *13*(2), 86–95. https://doi.org/10.37134/jpsmm.vol13.2.8.2023

Umam, K., & Susandi, A. D. (2022). Critical Thinking Skills: Error Identifications on Students' with APOS Theory. *International Journal of Evaluation and Research in Education*, *11*(1), 182–192. https://doi.org/10.11591/ijere.v11i1.21171

Wilkie, K. J. (2024). Coordinating Visual and Algebraic Reasoning with Quadratic Functions. *Mathematics Education Research Journal*, *36*, 27–63. https://doi.org/10.1007/s13394-022-00426-w

Yarman, Dwina, F., Murni, D., Yerizon, & Hevardani, K. A. (2024). Analysis of Concept Construction and Student Errors on the Topic of Double Integral Based on APOS Theory. *Jurnal Pendidikan Matematika*, *18*(3), 367–386. https://doi.org/10.22342/jpm.v18i3.pp367-386